Snow and sausages

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Dressed like extras for a cheap remake of *Scott of the Antarctic*, a select group braved their way through the January snow to Florence Boot Hall at the University of Nottingham to participate in the annual M500 Winter Recreational Mathematics Weekend. Mel and Angela guided us through a selection of 'investigations' which formed the theme of the weekend. These were simply-worded problems that lend themselves to 'extensions'. There were no 'right' answers, and no limits to how far you chose to take the investigation. My thanks to Mel Starkings and Angela Allsopp for an entertaining and challenging weekend, to Rob Rolfe who organized and presented our Friday Night Quiz, and to Diana Maxwell for arranging everything and for ensuring that all ran smoothly.

On the Friday evening, Mel set us thinking about the mathematics of sausages. Suppose that you have a string of sausages (these are, of course, mathematical sausages and so all are of equal length); then, in general, the string could be arranged in a triangular shape. Given a known number of sausages in the string, then the sausage number is the maximum number of different (non-congruent) triangles that you can construct from a given string. If we let N be the number of sausages, and $f(N)$ be the sausage number, then what is $f(42)$, and $f(2010)$? Is there a general formula?

A little bit of thought reveals that the sausages are not essential to the problem (although they are essential to the cooked breakfast that we enjoyed in the Dining Hall on both mornings, in my opinion). What we are considering is triangles with three integer sides. Let the three sides be a, b and c. Then

$$a + b + c \ = \ N.$$

If we are only considering real triangles (i.e. each triangle encloses a real, finite area), then each side must be at least 1 unit long, so

$$a, b, c, N \ \in \ \mathbb{Z}^+ \quad \text{and} \quad N \geq 3.$$

Now, without loss of generality, let us assume that the lengths of the three sides are such that $a \geq b \geq c$. Then

$$a < b + c, \qquad a < N - a, \qquad a < \frac{N}{2}.$$

And the other two sides will be less than, or equal to, a. Using integer arithmetic, we can define the upper bound on a as

$$A_U \ = \ (N - 1) \div 2.$$

The lower bound on a will occur when $a = b = c$ (if that is possible). In integer arithmetic this can be expressed as

$$A_L \;=\; (N+2) \div 3.$$

And so we can say that

$$a \;\in\; \{A_L \cap A_U\}.$$

The number of elements in the set of a is

$$N_a \;=\; A_U - A_L + 1.$$

Each of the elements in the set of a will appear in at least one triangle, so N_a represents a lower bound on $f(N)$.

Now considering the 'next longest' side, b, its upper bound must be a (otherwise it would be longer than a and so a would not be the longest side), so

$$B_U \;=\; a.$$

The lower bound on b will occur when $b = c$ (if that is possible). In integer arithmetic this can be expressed as

$$B_L \;=\; (N-a+1) \div 2.$$

And similarly

$$b \;\in\; \{B_U \cap B_L\}.$$

So, for a given N, we can find the largest and smallest values of a. Thus a is integer and will take on each of the integer values between A_L and A_U. For each value of a, we can find the largest and smallest values of b. Indeed, b is integer and will take on each of the integer values between B_L and B_U. And for each combination of a and b, we can find c as $c = N-(a+b)$, and so we can define each possible triangle in turn, and then count them to find the sausage number, $f(N)$. Incidentally, because of the way that the sides of the triangles are found, they must all be mutually non-congruent. The discussion above does not lead to a general formula for $f(N)$, but is the basis of an algorithm for finding $f(N)$ for any given N.

Using this algorithm, the values of $f(N)$ for N between 3 and 13, with intermediate values, are shown in the big table on the next page. And values of $f(N)$ for N between 3 and 32 are on the little table below it.

N	A_U	A_L	B_U a	B_L	b	c	$f(N)$
3	1	1	1	1	1	1	1
4	1	2					0
5	2	2	2	2	2	1	1
6	2	2	2	2	2	1	1
7	3	3	3	2	2 3	2 1	2
8	3	3	3	3	3	2	1
9	4	3	3 4	3 3	3 3 4	3 2 1	3
10	4	4	4	3	3 4	3 2	2
11	5	4	4 5	4 3	4 3 4 2	3 3 2 1	4
12	5	4	4 5	4 4	4 4 5	4 3 2	3
13	6	5	5 6	4 4	4 5 4 5 6	4 3 3 2 1	5

N	$f(N)$	N	$f(N)$	N	$f(N)$	N	$f(N)$	N	$f(N)$
3	1	9	3	15	7	21	12	27	19
4	0	10	2	16	5	22	10	28	16
5	1	11	4	17	8	23	14	29	21
6	1	12	3	18	7	24	12	30	19
7	2	13	5	19	10	25	16	31	24
8	1	14	4	20	8	26	14	32	21

Applying this algorithm to the case where $N = 42$ we can find that

$$f(42) = 37.$$

This is about the limit if calculating sausage numbers by this method by hand. As N increases, so the number of individual calculations needed to find $f(N)$ increases. I wrote a small computer program based on this method to find $f(N)$. This enabled me to answer the question, What is $f(2010)$? And the answer is . . . 84,169.

This is the 'sledgehammer' approach to maths and is frowned upon by Luddite purists. So, is there a general formula for $f(N)$?
