

Tricks of Mental Arithmetic

Mel amazed us with his knowledge of all squares ending in 5.
As examples he used:-

$$15^2 = 225$$

$$25^2 = 625$$

$$35^2 = 1225$$

$$45^2 = 2025$$

$$75^2 = 5625$$

$$85^2 = 7225$$

He then let us into a secret; there is of course a trick to use in calculating these squares. If you take the first digit and multiply it by one number higher, then place 25 on the end you have the answer!

E.g. To calculate 35^2 $3 \times 4 = 12$ putting 25 on the end gives 1225 which is the answer.

He then went on to explain how other multiplications could be done by utilizing algebraic techniques.

$$30^2 = 900 \quad (\text{as most of us could work out})$$

$31^2 = 961$ Which he could quickly calculate using $(n + 1)(n + 1)$ with $n = 30$.

$$(n+1)(n+1) = n^2 + 2n + 1, \text{ which can be rearranged to } n^2 + n + (n+1)$$

$$\text{hence, } 31^2 = 30^2 + 30 + 31 = 961.$$

This technique can be used to calculate other squares with different values of n for example when calculating 36^2 , n would equal 35 and the techniques explained earlier for calculating squares ending in 5 would be utilized.

$$36^2 = 35^2 + 35 + 36 = 1296$$

Calculations such as 39^2 can be done by using $(n-1)(n-1)$, with $n = 40$

$$(n-1)(n-1) = n^2 - 2n + 1, \text{ which can be rearranged to } n^2 - n - (n-1)$$

$$\text{hence } 39^2 = 40^2 - 40 - 39 = 1521$$

42^2 can be calculated using $(n+2)^2$ with $n = 40$.

$$(n+2)(n+2) = n^2 + 4n + 4, \text{ which can be rearranged to } n^2 + n + (n+1) + (n+1) + (n+2),$$

$$\text{hence } 42^2 = 40^2 + 40 + 41 + 41 + 42 = 1764$$

Mel went on to explain that many people teach multiplication by use of the difference of two squares. This enables people to mentally calculate such sums as

$$77 \times 73$$

$$78 \times 72$$

$$79 \times 71$$

$$32 \times 28$$

$$62 \times 58$$

$$63 \times 57$$

A closer look at the numbers to be multiplied reveals that they are equidistant from a number that is easily squared. A calculation using $(n + a)(n - a) = n^2 - a^2$ can then be carried out mentally, where a is the difference between the easily squared n and the numbers to be multiplied.

The above sums were therefore calculated as follows:-

$$\begin{aligned}77 \times 73 &= 75^2 - 2^2 = 5625 - 4 = 5621 \\78 \times 72 &= 75^2 - 3^2 = 5625 - 9 = 5616 \\79 \times 71 &= 75^2 - 4^2 = 5625 - 16 = 5609\end{aligned}$$

$$\begin{aligned}32 \times 28 &= 30^2 - 2^2 = 900 - 4 = 896 \\62 \times 58 &= 60^2 - 2^2 = 3600 - 4 = 3596 \\63 \times 57 &= 60^2 - 3^2 = 3600 - 9 = 3591\end{aligned}$$

Mathematics of Entertaining

Leaving behind the tricks for multiplying we then went on to discuss the mathematics of entertaining, i.e. you turn up for a dinner party and you are given a list of rules as to who you are not allowed to sit next to!

Everyone agreed that there were $n!$ ways of arranging n things, so if 7 people were sat at a table there were $7!$ ways or arrangements of sitting people down.

We then went on to consider how many ways there were of de-arranging everyone, so that no one could stay in the same place.

It was agreed $D(1) = 0$ (if there is only one person they can't change places with anyone)

$$D(2) = 1$$

$$D(3) = 2$$

$$D(4) = 9$$

$$D(5) = 44$$

$D(4)$ was calculated using 4 volunteers, and several groups came up with 44 for $D(5)$, and Mel then recognised that this was the correct figure. He left the group with the problem of finding a formula to calculate $D(n)$.

Two iterative formulae were found to find $D(n)$

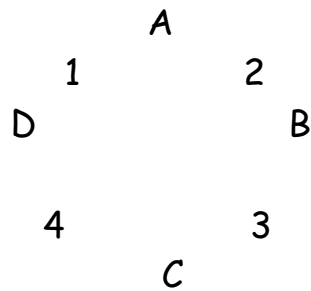
$$D_n = nD_{n-1} + (-1)^n \text{ with } D_1 = 0$$

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

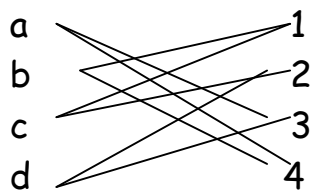
Mel's formula is a very surprising result

$$D_n = \text{The integer part of } \frac{n! + 1}{e}$$

Continuing on the theme of entertaining we looked at the arrangements of 4 couples sat at the table for dinner. The rule is no one can sit next to their partner and we assumed the women were all sat down.



(Capital letters are women, small case men, and numbers positions)



Many of you recognised this as a bipartite graph. A matching augmentation algorithm can now be carried out to find an alternating path to find a maximum matching.

Three people were then invited to link arms to establish if it is possible for three people to be linked, so that if any one of the three let go then all three would drift apart.

After much trying and laughter it was decided that it was possible, but unfortunately I am unable to reproduce a drawing to illustrate this! However Mel has given me the following explanation

The algorithm for three people linking arms is:

Two hold hands with themselves.....to provide the first two loops and one then stands alongside the first with their loop inside the loop.

The third person then faces the second threading the top arm through and then completes the loop.....

I hope these notes bring back happy memories of the weekend and we hope to see you again next year.

Angela